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ON THE DEFORMATION OF TWO DROPLETS IN A QUASISTEADY STOKES FLOW

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Abstract—A theoretical method is given for the determination of the shape of two drops (bubbles) moving with constant velocities parallel to their line of centres in a quiescent viscous fluid. The Reynolds numbers for the motions within the fluids are assumed to be sufficiently small that the equations governing these motions are quasisteady Stokes' equations. It is also assumed that the maximum deviation of the interfaces from spherical form is small when compared with the radius of the "equivalent" spherical drop. The paper deduces the first-order pressure distribution exterior and interior to the droplets. Effects of fluid viscosities, capillary numbers and distances between the droplets are taken into account. Special attention is paid to the influence of a solid plane or solid sphere on the shape of a drop (bubble) approaching or receding away from the solid boundary. The obtained solutions may serve as a first iteration of an iterative procedure for determining more accurate flow fields, taking into account the deviation from sphericity of the deformed particles.

1. INTRODUCTION

The area of bubble and drop phenomena has continued to receive much attention from investigators in the fields of chemical and biomedical engineering and science. The majority of the works are fundamental in nature, but permit one to develop rational understanding of the many industrial systems and processes such as sedimentation, gas-liquid contacting, flotation, fermentation liquid-liquid extraction, spray drying etc. An excellent review summarising the current state of knowledge in this field in the case of low Reynolds number flow and pointing out the utility of such knowledge in applications is given by Brenner (1971).

The translation of a single liquid sphere was first treated independently by Rybczynski (1911) and Hadamard (1911). Interfacial tension acting on the interface between the immiscible fluids tends to deform it. If the motion is sufficiently slow or the particle sufficiently small, the droplet will in the first approximation be spherical.

The investigations of the shape of a viscous drop settling through a fluid appear to be the most detailed treatment of a free surface. These studies allow the fluids on both sides of the free surface to exert shear forces as well as pressure forces.

The problems associated with the shape of droplet undergoing distortion, when inertia effects are no longer negligible, were discussed by Taylor & Acrivos (1964) and Pan & Acrivos (1968).

The motion of solid and liquid particles in a viscous medium at low Reynolds numbers is usually affected by the presence of rigid walls, free surfaces and adjacent solid particles or droplets.

An approximate solution to the motion of a solid spherical particle approaching a solid plane for the case where the radius of the sphere is small compared with the instantaneous distance of its midpoint from the plane was provided by Lorentz (1907). An exact solution of the Stokes' equations for the steady axisymmetric motion of a viscous fluid, posed when two spheres translate with equal velocities has been obtained by Stimson & Jeffery (1926) using bipolar coordinate transformation.

The first investigation of the behavior of a drop in shear flow was made by G. I. Taylor (1932), who stated the boundary conditions that must be satisfied at the drop surface, and found the velocity fields outside and inside a drop kept spherical by surface tension. The solution contained discontinuity in the normal stress at the drop surface. Taylor later (1934) found the deformation of the drop by equating the discontinuity in normal stress over the drop that was spherical with the pressure arising from curvature and surface tension over the deformed drop. Cox (1969) considered the deformation of a drop in a time-dependent shear flow. The solution to the limiting case problem of a solid particle approaching a solid plane or free surface has been presented independently by Brenner (1961) and Maude (1961). Using Lamb's solution of the Stokes' equation Chaffey *et al.* (1965) found the radial migration of a liquid drop suspended in a viscous fluid between counter-rotating discs or near a plane wall. An analysis of the motion of a spherical droplet settling towards a plane surface or interface between two immiscible viscous fluids at low Reynolds numbers has been given by Bart (1968). Rushton & Davies (1973, 1978) and Haber *et al.* (1973) considered the relative motion of two fluid spheres falling along their line of centres.

Recently, Leal *et al.* (1979, 1980, 1982a,b) have presented approximate analytical and numerical solutions to the problem of the motion of a solid sphere towards a nondeformable or deformable interface. O'Neill & Ranger (1983) have proposed an approximate solution to the problem posed when a rigid sphere normally approaches an interface between two immiscible viscous fluids.

The present work is an investigation of the shape of a drop moving towards or away from a solid plate or another deformable drop. The fluids involved are homogeneous, incompressible, Newtonian and have constant physical properties. The Reynolds numbers for the motions within the fluids will be assumed to be sufficiently small that the equations governing the motions are the quasisteady Stokes' equations. The boundary conditions are formulated on the hypothesis of no relative motion at the fluid-fluid interfaces and on the condition that the tangential stresses are continuous through the interfaces. For normal stresses we have supposed that any discontinuity in the normal stress at an interface is balanced by surface tension forces. The purpose of the paper is to find approximately the shape of the drops (bubbles) when the motion is slow and the maximum deviation of the drop interface from spherical form is small compared with the sphere radius. For this purpose the first-order pressure distribution exterior and interior to the droplets have been calculated. This does not appear to have been done before.

2. FORMULATION OF THE PROBLEM

Let us consider two liquid droplets moving along their line of centres with terminal velocities $\overline{U}_a$ and $\overline{U}_b$, respectively, in an unbounded quiescent fluid. It is assumed that the droplets move with slow relative velocity, such that the inertia terms in the equations of motion can be neglected.

The governing field equations are as follows:

i) For the interior of droplet a

$$\mu_a \nabla^2 \overline{V}_a = \nabla P_a,$$
$$\nabla \cdot \overline{V}_a = 0.$$

ii) For the interior of droplet b

$$\mu_b \nabla^2 \overline{V}_b = \nabla P_b,$$
$$\nabla \cdot \overline{V}_b = 0.$$

iii) For the field exterior to the droplets

$$\mu_e \nabla^2 \overline{V}_e = \nabla P_e,$$
$$\nabla \cdot \overline{V}_e = 0.$$

where $\overline{V}$ and P are the velocity and the hydrodynamic fluid pressure including gravitational body force potential if appropriate and μ is the fluid viscosity.

The boundary conditions are as follows:

a) The tangential and normal components of the velocity vectors inside and outside of the droplets are continuous on the interface

$$\overline{V}^{(e)} \cdot \overline{\tau}^0 = \overline{V}^{(i)} \cdot \overline{\tau}^0,$$
$$\overline{V}^{(e)} \cdot \overline{n}^0 = \overline{U}_j \cdot \overline{n}^0 \quad (j = a, b)$$
$$\overline{V}^{(i)} \cdot \overline{n}^0 = \overline{U}_j \cdot \overline{n}^0,$$

where $\overline{n}^0$, $\overline{\tau}^0$ denote, respectively, a unit vector normal to and tangent to a meridian curve of the interface; $\overline{U}_a$ and $\overline{U}_b$ are the velocities of the drops; the subscripts e and i indicate a property exterior or interior to the droplets.

b) The tangential components of the stress vectors of the fluids interior and exterior to the droplets are continuous through the interface

$$P^{(e)}_{n\tau} = P^{(i)}_{n\tau}.$$

c) The normal component of the stress vectors have a discontinuity which is proportional to the product of the surface tension σ and the mean curvature of the interface (Scriven 1960)

$$P^{(e)}_{nn} - P^{(i)}_{nn} = \sigma_j\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

where R_1 and R_2 are the principal radii of the deformed interface.

d) Far from the droplets the flow field is unperturbed and we can assume, without any loss of generality, that the velocity vector vanishes

$$\overline{V}^{(e)} \longrightarrow 0 \text{ far from the droplets.}$$

e) Inside of the droplets the velocity vector is finite

$$\overline{V}^{(i)} = O(1) \text{ inside of the droplets.}$$

For axisymetric motion the velocity components in cylindrical coordinates (τ, z, φ) may be expressed in terms of the Stokes' stream function ψ

$$V_\tau = \frac{1}{\tau}\frac{\partial\psi}{\partial Z}, \quad V_z = -\frac{1}{\tau}\frac{\partial\psi}{\partial\tau}, \quad V_\varphi = 0.$$

Hence the equations of the motion may be written as

$$E^4\psi = 0, \tag{1}$$

where

$$E^2 = \frac{\partial^2}{\partial\tau^2} - \frac{1}{\tau}\frac{\partial}{\partial\tau} + \frac{\partial^2}{\partial z^2}.$$

3. A METHOD FOR SOLUTION OF THE PROBLEM

The solution is initiated by assuming that the droplets are spherical and solving the flow fields. Subsequently, the geometry of the interface is solved for these flow fields. Let us denote the radii of the droplets by a and b. With b as a typical geometrical length U_b as a typical velocity and $\rho^{(e)}U_b^2$ as a typical pressure we will have the following dimensionless parameters, the Reynolds number of the problem $Re = \rho^{(e)}U_b b/\mu^{(e)}$, the Weber number for each droplet $We^a = \rho^{(e)}bU_b^2/\sigma_a$, $We^b = \rho^{(e)}bU_b^2/\sigma_b$ and $K_a = \mu_a^{(i)}/\mu^{(e)}$, $K_b = \mu_b^{(i)}/\mu^{(e)}$. Then for the dimensionless velocities of the droplets we shall have

$$V = \begin{cases} \pm \dfrac{U_a}{U_b} & \text{at } \eta = \eta_a \\ \pm 1 & \text{at } \eta = \eta_b, \end{cases}$$

where the sign depends on the direction of their movement; η_a and η_b are described further on.

Due to the specific geometry of the two fluid interfaces it is convenient to use bispherical coordinates (ξ, η, φ). We take the conformal transformation

$$z = \frac{c\, sh\eta}{ch\eta - \cos\xi}, \quad \tau = \frac{c \sin\xi}{ch\eta - \cos\xi},$$

where $0 \leq \xi \leq \pi$, $-\infty < \eta < \infty$, $0 \leq \varphi < 2\pi$ and c is a positive constant. The coordinate surfaces $\eta = \eta_a > 0$ and $\eta = \eta_b < 0$ are nonintersecting spheres whose centres lie along the z axis and $\eta_a = cth^{-1}\, d_a/c$, $\eta_b = ch^{-1}\, d_b$. Here d_a and d_b are the dimensionless distances from sphere centres to the radial plane $z = 0$. The value $\eta_a = 0$ corresponds to a sphere of infinite radius and is equivalent to the entire plane $z = 0$.

The relationships between the velocity components V_ξ, V_η and the stream function ψ are

$$V_\eta = -\frac{(ch\eta - \beta)^2}{c^2}\frac{\partial\psi}{\partial\beta},$$

$$V_\xi = -\frac{(ch\eta - \beta)^2}{c^2 \sin\xi}\frac{\partial\psi}{\partial\eta},$$

where $\beta = \cos\xi$. A solution of [1] in bipolar coordinates, suitable for satisfying boundary conditions was given by Stimson & Jeffery (1926)

$$\psi = (ch\eta - \beta)^{-3/2}\sum_{n=1}^{\infty} U_n(\eta)\cdot V_n(\beta). \qquad [2]$$

Here

$$U_n(\eta) = A_n ch(n - \tfrac{1}{2})\eta + B_n sh(n - \tfrac{1}{2})\eta + C_n ch(n + \tfrac{3}{2})\eta + D_n sh(n + \tfrac{3}{2})\eta$$

and $V_n(\beta)$ is a function related to Legendre polinomials via the relation

$$V_n(\beta) = P_{n-1}(\beta) - P_{n+1}(\beta).$$

The constants A_n, B_n, C_n and D_n are determined from the boundary conditions a), b), d) and e).

3.1. *Calculation of the pressure*

Since our aim is to find the shape of the droplets we shall use the normal force balance

$$P_{\eta\eta}^{(e)} - P_{\eta\eta}^{(i)} = \frac{1}{We}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \qquad [3]$$

where

$$P^{(e)}_{\eta\eta} = -P^{(e)} - \frac{2}{\mathrm{Re}}\frac{ch\eta - \beta}{c}\left[\frac{\partial}{\partial\eta}\frac{(ch\eta-\beta)^2}{c^2}\frac{\partial\psi^{(e)}}{\partial\beta} - \frac{ch\eta-\beta}{c^2}\frac{\partial\psi^{(e)}}{\partial\eta}\right],$$
$$P^{(i)}_{\eta\eta} = -P^{(i)} - \frac{2K}{\mathrm{Re}}\frac{ch\eta - \beta}{c}\left[\frac{\partial}{\partial\eta}\frac{(ch\eta-\beta)^2}{c^2}\frac{\partial\psi^{(i)}}{\partial\beta} - \frac{ch\eta-\beta}{c^2}\frac{\partial\psi^{(i)}}{\partial\eta}\right] \qquad [4]$$

and $P^{(e)}$, $P^{(i)}$ are dimensionless pressures outside and inside of the droplets.

The pressure of the fluids exterior and interior to the droplets can be calculated from equations

$$\frac{\partial P^{(e)}}{\partial\eta} = -\frac{ch\eta-\beta}{c\,\mathrm{Re}}\frac{\partial(E^2\psi^{(e)})}{\partial\beta},$$
$$\frac{\partial P^{(e)}}{\partial\beta} = \frac{ch\eta-\beta}{c\,\mathrm{Re}(1-\beta^2)}\frac{\partial(E^2\psi^{(e)})}{\partial\eta}, \qquad [5]$$

$$\frac{\partial P^{(i)}}{\partial\eta} = -\frac{ch\eta-\beta}{c\,\mathrm{Re}}K\frac{\partial(E^2\psi^{(i)})}{\partial\beta},$$
$$\frac{\partial P^{(i)}}{\partial\beta} = \frac{ch\eta-\beta}{c\,\mathrm{Re}(1-\beta^2)}K\frac{\partial(E^2\psi^{(i)})}{\partial\eta}, \qquad [6]$$

where

$$E^2 = \frac{ch\eta-\beta}{c^2}\left\{\frac{\partial}{\partial\eta}\left[(ch\eta-\beta)\frac{\partial}{\partial\eta}\right] + (1-\beta^2)\frac{\partial}{\partial\beta}\left[(ch\eta-\beta)\frac{\partial}{\partial\beta}\right]\right\}.$$

According to [5] and [6] the pressure P is a harmonic function, i.e.

$$\nabla^2 P = 0$$

and can therefore be expressed in terms of Jeffery's (1912) solution of Laplace's equation in bipolar coordinates:

$$P^{(e)} = \frac{\sqrt{ch\eta-\beta}}{c^3\,\mathrm{Re}}\sum_{n=0}^{\infty}\left[\alpha_n^{(e)}\,ch\left(n+\frac{1}{2}\right)\eta + \beta_n^{(e)}\,sh\left(n+\frac{1}{2}\right)\eta\right]P_n(\beta) + \Pi^{(e)}, \qquad [7]$$

$$P^{(i)} = \frac{\sqrt{ch\eta-\beta}}{c^3\,\mathrm{Re}}K\sum_{n=0}^{\infty}\left[\alpha_n^{(i)}\,ch\left(n+\frac{1}{2}\right)\eta + \beta_n^{(i)}\,sh\left(n+\frac{1}{2}\right)\eta\right]P_n(\beta) + \Pi^{(i)}, \qquad [8]$$

where $\Pi^{(e)}$ and $\Pi^{(i)}$ are arbitrary constants. The coefficients $\alpha_n^{(e)}$, $\beta_n^{(e)}$, $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ are to be determined by making use of equations [5] and [6].

According to Stimson & Jeffery (1926)

$$E^2(\psi) = \frac{\sqrt{ch\eta-\beta}}{c^2}\sum_{n=1}^{\infty}\left\{V_n(\beta)\left[\frac{d^2U_n}{d\eta^2} - \frac{2\,sh\eta}{ch\eta-\beta}\frac{dU_n}{d\eta} + \frac{3}{4}\frac{ch\eta+3\beta}{ch\eta-\beta}U_n\right] + (1-\beta^2)U_n\left[\frac{d^2V_n}{d\beta^2} + \frac{2}{ch\eta-\beta}\frac{dV_n}{d\beta}\right]\right\}. \qquad [9]$$

In order to separate the variables in [9] one can use the following recurrence relations

$$(1-\beta)^2\frac{dV_n}{d\beta}=\frac{n(n-1)}{2n-1}V_{n-1}-\frac{(n+1)(n+2)}{2n+3}V_{n+1},$$

$$\beta V_n=\frac{n-1}{2n-1}V_{n-1}+\frac{n+2}{2n+3}V_{n+1},$$

$$(1-\beta^2)\frac{d^2V_n}{d\beta^2}+n(n+1)V_n=0.$$

Then after somewhat lengthy algebra, [9] can be written in the form

$$E^2(\psi)=\frac{1}{c^2\sqrt{ch\eta-\beta}}\sum_{n=1}^{\infty}\left[a_n ch\left(n+\frac{1}{2}\right)\eta+b_n sh\left(n+\frac{1}{2}\right)\eta\right]V_n, \qquad [10]$$

where

$$a_n=-(2n-1)A_n+(2n+3)C_n+\frac{2n(2n+3)}{2n+1}A_{n+1}-\frac{2(n+1)(2n-1)}{2n+1}C_{n-1}$$

$$b_n=-(2n-1)B_n+(2_n+3)D_n+\frac{2n(2n+3)}{2n+1}B_{n+1}-\frac{2(n+1)(2n-1)}{2n+1}D_{n-1}$$

Substituting [7] and [8] into [5] and [6] and making use of [10] one can find

$$\alpha_n=\sum_{m=1}^{n-1}\left[\frac{2m+1}{m(m+1)}b_m+\frac{2n+1}{n}b_n\right]+\alpha_0,$$

$$\beta_n=\sum_{m=1}^{n-1}\left[\frac{2m+1}{m(m+1)}a_m+\frac{2n+1}{n}a_n\right]+\beta_0.$$

Furthermore, the coefficients $\alpha_0^{(e)}$ and $\beta_0^{(e)}$ are readily obtained by the condition that the pressure is finite at infinity.

$$\alpha_0^{(e)}=-\lim_{n\to\infty}\left[\sum_{m=1}^{n-1}\frac{2m+1}{m(m+1)}b_m^{(e)}+\frac{2n+1}{n}b_n^{(e)}\right],$$

$$\beta_0^{(e)}=-\lim_{n\to\infty}\left[\sum_{m=1}^{n-1}\frac{2m+1}{m(m+1)}a_m^{(e)}+\frac{2n+1}{n}a_n^{(e)}\right].$$

Let us consider the regions inside of the droplets. From the boundary conditions for the stream function it is easy to deduce that $P^{(i)}$ is finite as $\eta\to\pm\infty$ and is given by

$$P^{(i)}=\frac{\sqrt{ch\eta-\beta}}{c^3\,\mathrm{Re}}K\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n-1}\frac{2k+1}{k(k+1)}b_k^{(i)}+\frac{2n+1}{n}b_n^{(i)}\right]\times e^{\pm(n+1/2)\eta}P_n(\beta)+\Pi^{(i)}.$$

The sign "+" corresponds to $\eta<0$, and the sign "$-$" corresponds to $\eta>0$. If we take into account the gravitational body force we obtain

$$P^{(e)}=\frac{\sqrt{ch\eta-\beta}}{c^3\,\mathrm{Re}}\sum_{n=0}^{\infty}\left[\alpha_n^{(e)}ch\left(n+\frac{1}{2}\right)\eta+\beta_n^{(e)}sh\left(n+\frac{1}{2}\right)\eta\right]P_n(\beta)$$
$$-\frac{gb}{U_b^2}c\frac{sh\eta}{ch\eta-\beta}+\Pi^{(e)}, \qquad (11)$$

$$P^{(i)} = \frac{\sqrt{ch\eta - \beta}}{c^3 \mathrm{Re}} K \sum_{n=1}^{\infty} \alpha_n^{(i)} e^{\pm(n+1/2)\eta} P_n(\beta) - \frac{gb\gamma}{U_b^2} c \frac{sh\eta}{ch\eta - \beta} + \Pi^{(i)}, \qquad [12]$$

where

$$\gamma_{a,b} = \begin{cases} \dfrac{\rho_a^{(i)}}{\rho^{(e)}} \\ \dfrac{\rho_b^{(i)}}{\rho^{(e)}} . \end{cases}$$

3.2. *Equations for the shape of the droplets*

By using the equations [2], [4], [11], [12] and the expansions

$$\frac{1}{(ch\eta - \beta)^{1/2}} = \sqrt{2} \sum_{n=0}^{\infty} e^{\pm(n+1/2)\eta} P_n(\beta),$$

$$\frac{sh\eta}{(ch\eta - \beta)^{3/2}} = \mp \sqrt{2} \sum_{n=0}^{\infty} (2n+1)\, e^{\pm(n+1/2)\eta} P_n(\beta),$$

we obtain

$$\begin{aligned} P_{\eta\eta}^{(e)} - P_{\eta\eta}^{(i)} \Big|_{\eta=\eta_{a,b}} = \frac{\sqrt{ch\eta - \beta}}{c^3 \mathrm{Re}} \Bigg\{ \sum_{n=0}^{\infty} \Bigg[& K\alpha_n^{(i)} e^{\pm(n+1/2)\eta} \\ & - \alpha_n^{(e)} ch\left(n + \frac{1}{2}\right)\eta - \beta_n^{(e)} sh\left(n + \frac{1}{2}\right)\eta \\ & - \frac{gb}{U_b^2} c^4 \mathrm{Re}(1 - \gamma) \sqrt{2}\, [\pm(2n+1)]\, e^{\pm(n+1/2)\eta} \Bigg] P_n(\beta) \\ & + 2(1 - K) \Bigg\{ \sum_{n=1}^{\infty} (2n+1) \left(\frac{1}{2} sh\eta\, U_n^{(e)} + ch\eta \frac{dU_n^{(e)}}{d\eta}\right) P_n(\beta) \\ & + \sum_{n=2}^{\infty} \left[-\frac{2n-1}{2} \frac{dU_{n-1}^{(e)}}{d\eta} \pm \frac{3}{4\sqrt{2}} Vc^2 \frac{n(n-1)}{2n-1} e^{\pm(n-3/2)\eta} \right] P_n(\beta) \\ & + \sum_{n=0}^{\infty} \Bigg[-\frac{2n+3}{2} \frac{dU_{n+1}^{(e)}}{d\eta} \pm \frac{3}{4\sqrt{2}} Vc^2 \Bigg(e^{\pm(n+5/2)\eta} \frac{(n+2)(n+1)}{2n+3} \\ & - \frac{2(2n+1)(n^2+n-1)}{2n-1)\,(2n+3)} e^{\pm(n+1/2)\eta} \Bigg) \Bigg] P_n(\beta) \Bigg\} \\ & - (\Pi^{(e)} - K\Pi^{(i)}) c^3 \mathrm{Re} \sqrt{2} \sum_{n=0}^{\infty} e^{\pm(n+1/2)\eta} P_n(\beta) \Bigg\} \Bigg|_{\eta=\eta_{a,b}} . \end{aligned} \qquad [13]$$

The shape of the fluid interfaces may be represented by

$$\eta^H = \eta_0 + H(\beta),$$

where $\max_\beta | H(\beta) | < 1$ and $\eta_0 = \eta_{a,b}$. Since $\max_\beta | H(\beta) | < 1$ then (Aris 1962) the sum of the principal radii of the curvature can be written in the form

$$\frac{1}{R_1} + \frac{1}{R_2} = \pm \frac{1}{c} \left\{ (ch\eta_0 - \beta)^3 \frac{d}{d\beta} \left[\frac{1 - \beta^2}{(ch\eta_0 - \beta)^2} \frac{dH}{d\beta} \right] + 2Hch\eta_0 + 2sh\eta_0 \right\} . \qquad [14]$$

Here the upper sign corresponds to the case when $\eta_0 > 0$ and the lower sign to $\eta_0 < 0$.

Upon introducing the obtained results in [13] and [14] into equation [3] one finds the following ordinary differential equation

$$\begin{aligned}
&\frac{1}{\mathrm{We}\cdot c}\left\{(ch\eta_0-\beta)^3\frac{d}{d\beta}\left[\frac{1-\beta^2}{(ch\eta_0-\beta)^2}\frac{dH}{d\beta}\right]+2Hch\eta_0\right\}\\
&=\frac{\sqrt{ch\eta-\beta}}{c^3\,\mathrm{Re}}\left\{\sum_{n=0}^{\infty}\left[K\alpha_n^{(i)}e^{\pm(n+1/2)\eta}-\alpha_n^{(e)}ch\left(n+\frac{1}{2}\right)\eta-\beta_n^{(e)}sh\left(n+\frac{1}{2}\right)\eta\right.\right.\\
&\left.+\frac{3c^4\,\mathrm{Re}}{2\sqrt{2}\pi}\frac{1}{\tau_0^3}F_D(\pm(2_n+1))e^{\pm(n+1/2)\eta}\right]P_n(\beta)\\
&+2(1-K)\left\{\sum_{n=1}^{\infty}(2n+1)\left(\frac{1}{2}sh\eta U_n^{(e)}+ch\eta\frac{dU_n^{(e)}}{d\eta}\right)P_n(\beta)\right.\\
&+\sum_{n=2}^{\infty}\left[-\frac{2n-1}{2}\frac{dU_{n-1}^{(e)}}{d\eta}\pm\frac{3}{4\sqrt{2}}Vc^2\frac{n(n+1)}{2n-1}e^{\pm(n-3/2)\eta}\right]P_n(\beta)\\
&+\sum_{n=0}^{\infty}\left[-\frac{2n+3}{2}\frac{dU_{n+1}^{(e)}}{d\eta}\pm\frac{3}{4\sqrt{2}}Vc^2\left[e^{\pm(n+5/2)\eta}\frac{(n+2)(n+1)}{2n+3}\right.\right.\\
&\left.\left.\left.-\frac{2(2n+1)(n^2+n-1)}{(2n-1)(2n+3)}e^{\pm(n+1/2)\eta}\right]\right]P_n(\beta)\right\}\\
&\left.-\left(\Pi^{(e)}-K\Pi^{(i)}-\frac{2sh\eta}{\mathrm{We}c}\right)c^3\,\mathrm{Re}\,\sqrt{2}\sum_{n=0}^{\infty}e^{\pm(n+1/2)\eta}P_n(\beta)\right\}\Bigg|_{\eta=\eta_0}.
\end{aligned} \qquad [15]$$

Here it was taken into account an overall force balance

$$\frac{gb(1-\gamma)}{U_b^2}=-\frac{4}{3\pi}\frac{1}{\tau_0^3}F_D,$$

where

$$\tau_0=\begin{cases}\dfrac{a}{b} & \text{at}\quad \eta=\eta_a\\[2mm] 1 & \text{at}\quad \eta=\eta_b\end{cases}$$

and $F_D=\text{drag}\,/b^2\rho^{(e)}U_b^2$ is the dimensionless drag on the drop. In our case (Stimson & Jeffery 1926)

$$F_D=\frac{2\pi\sqrt{2}}{\mathrm{Re}}\cdot\frac{1}{c}\sum_{n=1}^{\infty}(A_n\pm B_n+C_n\pm D_n),$$

where the upper sign corresponds to the case when $\eta_0>0$ and the lower sign to $\eta_0<0$.

The function $H(\beta)$ is to be determined from [15] subject to the following conditions

$$\int_{-1}^{1}\frac{H(\beta)\,d\beta}{(ch\eta_0-\beta)^3}=0\quad \text{at}\ \max_\beta|H(\beta)|\ll 1. \qquad [16]$$

$$\int_{-1}^{1}\frac{H(\beta)\,d\beta}{(ch\eta_0-\beta)^4}=0\quad \text{at}\ \max_\beta|H(\beta)|\ll 1. \qquad [17]$$

The equation [16] represents the condition that the characteristic length b has been set equal to the radius of the "equivalent" spherical drop while [17] represents the condition that mass centre of the drop is preserved.

A particular solution of the equation

$$(ch\eta_0 - \beta)^3 \frac{d}{d\beta}\left[\frac{1-\beta^2}{(ch\eta_0-\beta)^2}\frac{dH(\beta)}{d\beta}\right] + 2H(\beta)ch\eta_0 = 0 \qquad [18]$$

has the form

$$1 - \beta ch\eta_0. \qquad [19]$$

Hence we can write down the solution of [15] in the form

$$H(\beta) = \frac{\mathrm{We}}{\mathrm{Re}}\frac{1}{c^2}(ch\eta_0 - \beta)^{3/2}\left[A\frac{1-\beta ch\eta_0}{(ch\eta_0-\beta)^{3/2}} + \sum_{n=0}^{\infty} H_n P_n(\beta)\right], \qquad [20]$$

where A is an arbitrary constant and

$$H_n = H_n^{(1)} + \Pi H_n^{(2)}, \quad \Pi = \Pi^{(e)} - K\Pi^{(2)} - \frac{2sh\eta_0}{\mathrm{We}c}.$$

We will note that the sum

$$\frac{\mathrm{We}}{\mathrm{Re}}\frac{(ch\eta_0-\beta)^{3/2}}{c^2}\sum_{n=0}^{\infty} H_n P_n(\beta)$$

is a particular solution of [15] and the constants A, Π are determined by the conditions [16] and [17]. Substituting [20] into [16] and [17] we obtain for A and Π

$$\sum_{n=0}^{\infty}(H_n^{(1)} + \Pi H_n^{(2)})e^{\pm(n+1/2)\eta_0} = 0,$$

$$\sqrt{2}\sum_{n=0}^{\infty}(H_n^{(1)} + \Pi H_n^{(2)})(2n+1)e^{\pm(n+1/2)\eta_0} = -A.$$

Substitution of [20] into [15] and the equation of the coefficients of $P_n(\beta)$ yields the following recurrence equations to determine the coefficients H_n:

i) For $n = 0$

$$-\frac{1}{2}H_2 + ch\eta_0 H_1 - \left(\frac{5}{2} - 2ch^2\eta_0\right)H_0$$

$$= K\alpha_0^{(i)}e^{\pm\eta_0/2} - \alpha_0^{(e)}\,ch\frac{\eta_0}{2} - \beta_0^{(e)}\,sh\frac{\eta_0}{2} \pm \frac{3c^4\,\mathrm{Re}}{2\sqrt{2}\pi}\frac{1}{\tau_0^3}F_D e^{\pm\eta_0/2} \qquad [21]$$

$$+ 2(1-K)\left[-\frac{3}{2}\frac{dU_1^{(e)}}{d\eta} \pm \frac{1}{2\sqrt{2}}Vc^2(e^{\pm 5/2\eta_0} - e^{\pm\eta_0/2})\right] + \Pi e^{\pm\eta_0/2}.$$

ii) For $n = 1$

$$-\frac{3}{2}H_3 + 4ch\eta_0 H_2 - \frac{7}{2}H_1 + ch\eta_0 H_0$$

$$= K\alpha_1^{(i)}e^{\pm 3/2\eta_0} - \alpha_1^{(e)}\,ch\frac{3}{2}\eta_0 - \beta_1^{(e)}\,sh\frac{3}{2}\eta_0 \pm \frac{9c^4\,\mathrm{Re}}{2\sqrt{2}\pi}\frac{1}{\tau_0^3}F_D\,e^{\pm 3/2\eta_0} + 2(1-\mathrm{K}) \qquad [22]$$

$$\cdot\left\{3\left(\frac{sh\,\eta_0 U_1^{(e)}}{2} + ch\eta_0\frac{dU_1^{(e)}}{d\eta}\right) - \frac{5}{2}\frac{dU_2^{(e)}}{d\eta} \pm \frac{3}{4\sqrt{2}}Vc^2\left[\frac{6}{5}e^{\pm 7\eta_0/2} - \frac{6}{5}e^{\pm 3\eta_0/2}\right]\right\} + \Pi e^{\pm 3/2\eta_0}.$$

iii) For $n = 2$

$$-3H_4 + 9ch\eta_0 H_3 - \left(\frac{11}{2} + 4ch^2\eta_0\right)H_2 + 4ch\eta_0 H_1 - \frac{1}{2}H_0 = K\alpha_2^{(i)} e^{\pm 5/2\eta_0} - \alpha_2^{(e)} ch\frac{5}{2}\eta_0$$
$$- \beta_2^{(e)} sh\frac{5}{2}\eta_0 \pm \frac{5c^4 \mathrm{Re}}{2\sqrt{2}\pi}\frac{1}{\tau_0^3}F_D e^{\pm 5/2\eta_0} + 2(1-K)\cdot\left\{5\left(\frac{sh\eta_0}{2}U_2^{(e)} + ch\eta_0\frac{dU_2^{(e)}}{d\eta}\right)\right. \quad [23]$$
$$\left.- \frac{5}{2}\frac{dU_1^{(e)}}{d\eta} - \frac{7}{2}\frac{dU_3^{(e)}}{d\eta} \pm \frac{3}{4\sqrt{2}}Vc^2\left(2e^{\pm\eta_0/2} - \frac{30}{21}e^{\pm 5/2\eta_0} + \frac{12}{7}e^{\pm 7/2\eta_0}\right)\right\} + \Pi e^{\pm 5/2\eta_0}.$$

iiii) For $n \geq 3$

$$-\frac{(n+2)(n+1)}{4}H_{n+2} + (n+1)^2 ch\eta_0 H_{n+1}$$
$$-\left[\frac{n^2+(n+1)^2+9}{4} + (n-1)(n+2)ch^2\eta_0\right]H_n + n^2 ch\eta_0 H_{n-1}$$
$$-\frac{n(n-1)}{4}H_{n-2} = K\alpha_n^{(i)} e^{\pm(n+1/2)\eta_0} - \alpha_n^{(e)} ch\left(n+\frac{1}{2}\right)\eta_0 - \beta_n^{(e)} sh\left(n+\frac{1}{2}\right)\eta_0 \quad [24]$$
$$\pm\frac{3c^4 \mathrm{Re}}{2\sqrt{2}\pi}\frac{1}{\tau_0^3}F_D(2n+1)e^{\pm(n+1/2)\eta_0} + 2(1-K)\left\{(2n+1)\left(\frac{1}{2}sh\eta_0 U_n^{(e)} + ch\eta_0\frac{dU_n^{(e)}}{d\eta}\right)\right.$$
$$-\frac{2n-1}{2}\frac{dU_{n-1}^{(e)}}{d\eta} - \frac{2n+3}{2}\frac{dU_{n+1}^{e}}{d\eta} \pm \frac{3}{4\sqrt{2}}Vc^2\left[\frac{n(n-1)}{2n-1}e^{\pm(n-3/2)\eta_0}\right.$$
$$\left.\left.-\frac{2(2n+1)(n^2+n-1)}{(2n-1)(2n+3)}e^{\pm(n+1/2)\eta_0} + \frac{(n+2)(n+1)}{2n+3}e^{\pm(n+5/2)\eta_0}\right]\right\} + \Pi e^{\pm(n+1/2)\eta_0}.$$

3.3. *A solution of the recurrence equations*

On the base of the upper formulas and the following algorithm for the successive calculation of the coefficients $H_0, H_1, H_2, \ldots, H_n, \ldots$ it is possible to obtain the deformation of the droplets, which is the main purpose of the present paper.

For the sake of brevity let us denote the sum of the members which depend on the known function ψ and the expression which contains the unknown constant Π in the right side of [21]–[24] by $F\psi$ and $F\Pi$, respectively. Then we obtain the following set of recurrence equations for H_n

$$-\tfrac{1}{2}H_2 + ch\eta_0 H_1 - (\tfrac{5}{2} - 2ch^2\eta_0)H_0 = F\psi_0 + F\Pi_0, \quad [25]$$

$$-\tfrac{3}{2}H_3 + 4ch\eta_0 H_2 - \tfrac{7}{2}H_1 + ch\eta_0 H_0 = F\psi_1 + F\Pi_1 \quad [26]$$

$$-3H_4 + 9ch\eta_0 H_3 - (\tfrac{11}{2} + 4ch^2\eta_0)H_2 + 4ch\eta_0 H_1 - \tfrac{1}{2}H_0 = F\psi_2 + F\Pi_2 \quad [27]$$

$$-\frac{(n+2)(n+1)}{4}H_{n+2} + (n+1)^2 ch\eta_0 H_{n+1}$$
$$-\left[\frac{n^2+(n+1)^2+9}{4} + (n-1)(n+2)ch^2\eta_0\right]H_n \quad [28]$$
$$+ n^2 ch\eta_0 H_{n-1} - \frac{n(n-1)}{4}H_{n-2} = F\psi_n + F\Pi_n.$$

Since the function $H(\beta)$ in [20] is limited at $\beta = 1$ it follows that $\{H_n\}_{n\to\infty} \to 0$. Consequently the unknown coefficients H_n have to satisfy recurrence formulas [25]–[28] and the condition $\lim_{n\to\infty} H_n = 0$.

Let us consider the recurrence equation [28]. The right-hand side of this equation has an order $O(e^{-n|\eta_0|})$. After dividing [28] by $(n + 2)(n + 1)$ therefore for sufficient large $n \geq N$ shall have

$$-\tfrac{1}{4} H_{n+2} + ch\eta_0 H_{n+1} - (\tfrac{1}{2} + ch^2\eta_0)H_n + ch\eta_0 H_{n-1} - \tfrac{1}{4}H_{n-2} = 0. \qquad [29]$$

For analytic purposes it is convenient to separate the infinite set [25]–[28] into two parts

i) finite system for $n = 0, 1, 2, \ldots, N$

ii) infinite system for $n = N + 1, N + 2, N + 3 \ldots.$

Using the condition $\{H_n\}_{n=N+1}^{\infty} \rightarrow 0$ as $n \rightarrow \infty$ one can solve the equations [29] expressing the coefficients $H_{N-1}, H_N, H_{N+1}, H_{N+2}, \ldots$ by means of two constants a and b. Substituting $H_{N-1}, H_N, H_{N+1}, H_{N+2}$ into [25–28] we can calculate the constants a, b and the coefficients $H_0, H_1, H_2, \ldots, H_{N-2}$.

Since the coefficients of system [29] are constants the solution of this system will satisfy the equation:

$$X^4 - 4ch\eta_0 X^3 + (2 + 4ch^2\eta_0)X^2 - 4ch\eta_0 X + 1 = 0.$$

On solving this equation one finds

$$X_{1,2} = e^{\eta_0} \quad \text{and} \quad X_{3,4} = e^{-\eta_0}.$$

Consequently the solution of the system [29] has the form

$$H_n = (a + bn)e^{-n|\eta_0|} + (c + dn)e^{n|\eta_0|} \quad \text{at } n \geq N - 1.$$

Then, in view of the condition $H_n \rightarrow 0$, as $n \rightarrow \infty$ we obtain $d = c = 0$. On substituting H_{N+2}, H_{N+1}, H_N and H_{N-1} into [28] and solving the resulting algebraic equation we find H_{N-2} as a function of the constants a and b. Proceeding in this manner we can get from [27] and [28] all the coefficients $H_{N-3}, H_{N-4}, \ldots, H_2, H_1, H_0$. Furthermore, the supplementary unknowns a and b are found from [25] and [26]. The equations [25] and [26] form a linear-dependent system but since we seek a particular solution we can put $a = 0$.

It is noteworthy that in the same way one can find a solution of the homogeneous recurrence equations [25]–[28], which will be a solution for the homogeneous differential equation [18]. Comparing this approximate solution with exact solution [19] we obtain a concurrence to 10^{-6}, which shows that the proposed method has a good accuracy and is of a great promise.

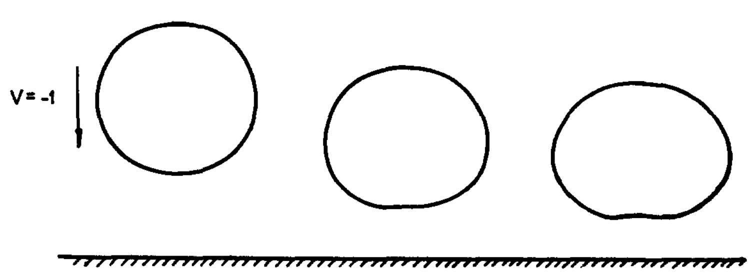

Figure 1. Liquid drop approaching a solid plane for $K = 0.5$, We/Re $= 0.1$, and different values of the separation distances: $d = 2$, $d = 1.5$, $d = 1.3$.

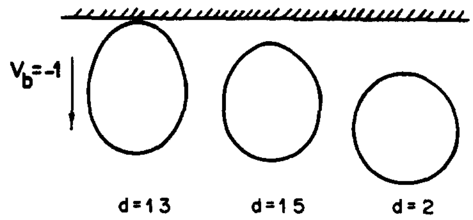

Figure 2. Liquid drop receding away from the solid plane for $K = 0.5$, We/Re = 0.1 and different values of the separation distances: $d = 1.3$, $d = 1.5$, $d = 2$.

4. NUMERICAL RESULTS FOR DEFORMATION OF THE FLUID PARTICLES

To illustrate the usefulness of our general solution given in the previous section we shall give some results in three different cases:

4.1. *The shape of a drop moving towards or away from a solid plane*

Consider, at first, a particular case of a drop approaching or receding away from a solid plane. We recall that fluid particles at low Reynolds number in infinite media tend to be spherical. Figure 1 represents the wall influence over the shape of a drop approaching a solid plane for $K = 0.5$, We/Re = 0.1 and different values of the separation distances: $d = 2$, $d = 1.5$ and $d = 1.3$. It is seen that as the separation distance d decreases, the presence of the wall causes a droplet deformation. With decreasing d the shape of a large part of the drop interface approaches a spherical form, while the front region becomes flat and a "dimple" between the fluid particle and the solid plane comes into view. Figure 2 shows the wall influence over the shape of a drop receding away from the solid plane for $K = 0.5$, We/Re = 0.1 and different values of the separation distances: $d = 1.3$, $d = 1.5$ and $d = 2$. When the fluid particle is moving away from a solid plane and the separation distance between them is

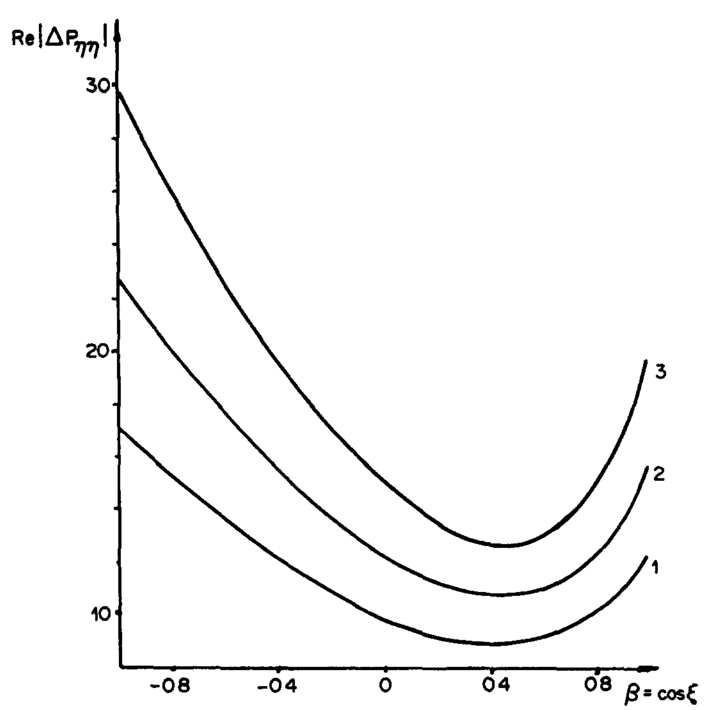

Figure 3. The distribution of $Re\,|\Delta P_{\eta\eta}|$ on the drop (bubble) interface for $d_b = 1.5$ and different values of the viscosity ratio: (1) $K = 0$, (2) $K = 1$, (3) $K = 10$.

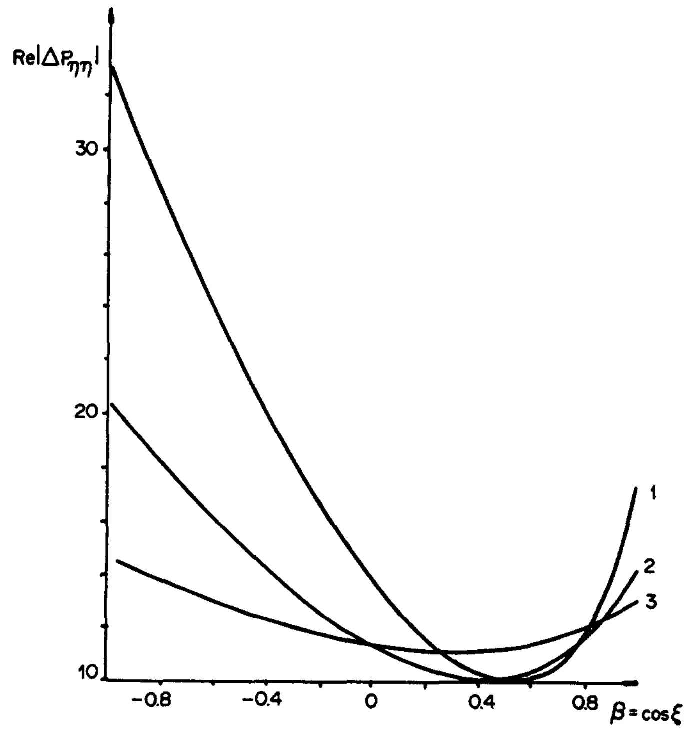

Figure 4. The distribution of Re $|\Delta P_{\eta\eta}|$ on the drop (bubble) interface for $K = 0.5$ and different values of the separation distances: (1) $d_b = 1.3$, (2) $d_b = 1.5$, (3) $d_b = 2$.

small an elogation is occurring in the vertical direction to yield approximately prolate ellipsoid shape. It is evident that in this case there is a sensitive influence of the normal stress differences, $\Delta P_{\eta\eta}$, pulling the drop downwards. The distribution of $|\Delta P_{\eta\eta}|$ on the drop (bubble) interface is given on figure 3 for $d = 1.5$ and different values of the viscosity ratio: $K = 0$, $K = 1$ and $K = 10$. It is seen that the normal stress differences increase as the viscosity ratio increases. Figure 4 illustrates the relationship between $|\Delta P_{\eta\eta}|$ and $\beta = \cos\xi$ over the drop interface for $K = 0.5$, and different values of the separation distances: $d = 1.3$, $d = 1.5$ and $d = 2$.

One can see that the normal stress differences $|\Delta P_{\eta\eta}|$ decrease as the separation distances increase. They take maximum values over the front part of the drop and minimum values over its side part. When the drop is moving towards the solid plane they are directed

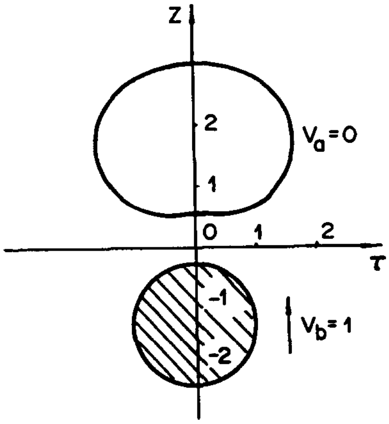

Figure 5. Solid spherical particle approaching a deformable drop for $K_a = 1.5$, We/Re $= 0.16$, $\tau_a = 1.5$, $d_a = 1.7$ and $\tau_b = 1$, $d_b = 1.5$.

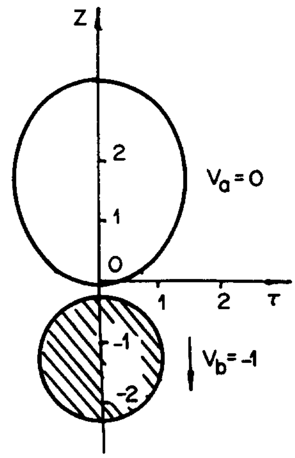

Figure 6. Solid spherical particle receding away from a deformable drop for $K_a = 1.5$, We/Re = 0.12, $\tau_a = 1.5$, $d_a = 1.7$ and $\tau_b = 1$, $d_b = 1.3$.

inward to the particle centre and when the drop is receding away from the solid plane they are directed outward to the particle, i.e. in the first case normal stress differences contract the drop and in the second one they stretch the fluid particle.

For assigned values of d and K the sign of the normal stress differences is changed depending on the movement direction of the drop.

4.2. *A solid spherical particle approaching or receding from a deformable drop (bubble)*

When a solid particle is approaching or receding from the deformable fluid particle its shape is changing because the excess pressure inside the drop is increasing. In figure 5 we demonstrate the influence of a solid sphere approaching the drop over the shape of the fluid particle for $\tau_b = 1$, $d_b = 1.5$ and $K_a = 1.5$, We/Re = 0.16, $\tau_a = 1.5$, $d_a = 1.7$.

The results for the interaction between a moving solid sphere and a standing drop at $\tau_a = 1.5$, $d_a = 1.7$, $K_a = 1.5$, We/Re = 0.12 and $\tau_b = 1$, $d_b = 1.3$ are shown in figure 6. Figure 7 presents how a settling solid sphere influences over the shape of a rising bubble ($\tau_a = 1$, $d_a = 1.19$, $\tau_b = 1$, $d_b = 1.19$, $K_b = 0$, We/Re = 0.11). It is interesting to note that the settling particle sucks up the rising bubble and changes its shape greatly.

4.3. *The relative motion of two deformable fluid particles*

As must be expected (Taylor & Acrivos, 1964), when the distance between the droplets tends to infinity there is no deformation of the fluid-liquid interfaces for low Reynolds steady

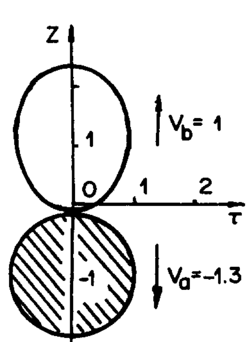

Figure 7. A relative motion of a solid particle and a bubble in opposite directions for $K_a = 0$, We/Re = 0.1, $\tau_a = 1$, $d_a = 1.2$ and $\tau_b = 1$, $d_b = 1.2$.

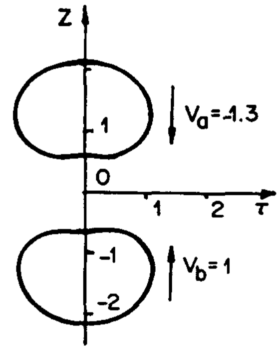

Figure 8. A relative motion of two deformable droplets for $K_a = 1$, $\mathrm{We}^a/\mathrm{Re} = 0.3$, $\tau_a = 1$, $d_a = 1.34$ and $K_b = 0.5$, $\mathrm{We}^b/\mathrm{Re} = 0.28$, $\tau_b = 1$, $d_b = 1.34$.

flow. For droplets travelling in the opposite direction and approaching each other there are two dimples. (See figure 8, where $\tau_a = 1$, $d_a = 1.34$, $\mathrm{We}^a/\mathrm{Re} = 0.3$, $K_a = 1$ and $\tau_b = 1$, $d_b = 1.34$, $\mathrm{We}^b/\mathrm{Re} = 0.28$, $K_b = 0.5$). The dimple formation mechanism is not well understood. It is complicated because of a coupling of bulk and interfacial fluid dynamics and because the drop's initial kinetic energy should be taken into account when describing the interfacial tension gradient mathematically. Unfortunately, the suggested theory does not permit to obtain quantitative results for the shape of two approaching drops when the separation distance between them is so small that the hydrodynamic film is formed. (Lin & Slattery, 1982.) In figure 9 we plot the shape of two fluid particles moving away from each other for $\tau_a = 1.5$, $d_a = 1.9$, $\mathrm{We}^a/\mathrm{Re} = 0.24$, $K_b = 1.5$ and $\tau_b = 1$, $d_b = 1.55$, $\mathrm{We}^b/\mathrm{Re} = 0.36$, $K_b = 1.5$.

It is seen from [20] that the shape of the fluid particles depends explicitly on the capillary number

$$C_a = \frac{\mathrm{We}}{\mathrm{Re}} = \frac{U_b \mu^{(e)}}{\sigma}.$$

One notes that for separation distances of order O [1], when interfacial tension σ decreases the degree of the drop deformations increases and for large σ the drop shape is nearly spherical.

At the same time figure 10 shows that there are such values of the given parameters ($\tau_a = 1.5$, $d_a = 1.7$, $\mathrm{We}^a/\mathrm{Re} = 0.54$, $K_a = 0.5$ and $\tau_b = 1$, $d_b = 1.34$, $\mathrm{We}^b/\mathrm{Re} < 300$, $K_b = 15$) that

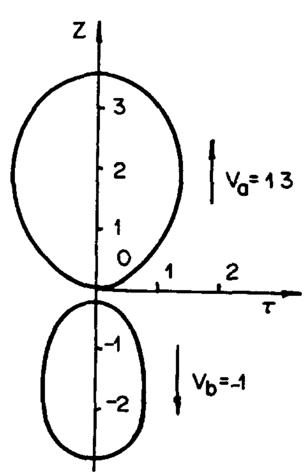

Figure 9. A relative motion of two deformable droplets for $K_a = 0$, $\mathrm{We}^a/\mathrm{Re} = 0.24$, $\tau_a = 1.5$, $d_a = 1.9$ and $k_b = 1.5$, $\mathrm{We}^b/\mathrm{Re} = 0.36$, $\tau_b = 1$, $d_b = 1.55$.

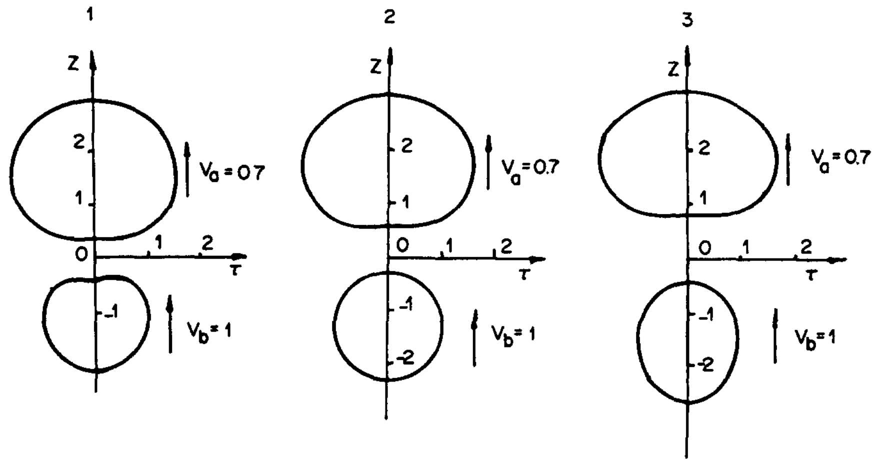

Figure 10. Two fluid drops moving in the same direction for $K_a = 0.5$, $\tau_a = 1.5$ and $K_b = 1.5$, $\tau_b = 1$, but for different values of d_a and d_b; 1) at $d_a = 1.6$, $d_b = 1.2$, $We^a/Re = 0.28$ and $We^b/Re = 1.6$; 2) at $d_a = 1.75$, $d_b = 1.34$, $We^a/Re = 0.54$ and any value of We^b/Re; 3) at $d_a = 1.9$, $d_b = 1.5$, $We^a/Re = 0.91$ and $We^b/Re = 1.6$.

the flow around the second drop may be like a drop in Stokes' unbounded flow, i.e. without any drop deformation.

Figure 11 presents the shape of two droplets moving in the same direction for $\tau_a = 1$, $d_a = 1.19$, $We^a/Re = 0.98$, $K_a = 0.5$ and $\tau_b = 1$, $d_b = 1.19$, $We^b/Re = 0.98$, $K_b = 0.5$. In this case the first drop sucks up the second one and as a result they have completely different shapes.

All pictorial material shows that the degree of deformation increases as the viscosity ratio K decreases. Finally we shall note that the driving force in all considered systems is the gravitational body force F, which depends on the gravitational accelaration g, the density ratios of the considered fluids $\gamma_{a,b}$ and so on.

5. CONCLUSIONS

A general first-order theory for determination of the shape of two droplets moving in a quasisteady Stokes' flow is presented. From a known stream function (Rushton & Davies 1973) the pressure distribution exterior and interior to the droplets has been calculated. The resulting solutions may be employed to predict the shape of the drops as a function of the viscosity ratios and the separation of the particles. When the distances of separation between two approaching or receding particles are not large the deformation takes place nevertheless whether the particles are gas bubbles or liquid droplets. In addition the proposed method

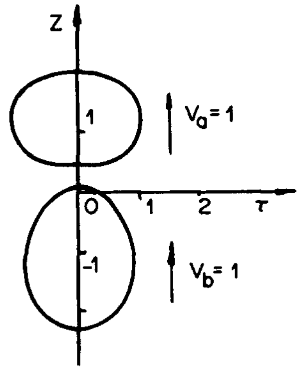

Figure 11. Two fluid drops moving in the same direction at $K_a = 0.5$, $We^a/Re = 0.98$, $\tau_a = 1$, $d_a = 1.2$ and $K_b = 0.5$, $We^b/Re = 0.98$, $\tau_b = 1$, $d_b = 1.2$.

allows to calculate the shape of a deformable drop (bubble) in some limiting cases. Two special cases may be worthy to mention in this respect. These are a drop (bubble) approaching or receding away from a solid plane and the relative motion of a drop (bubble) and a solid spherical particle.

It has been shown that when a fluid particle is moving away from a solid plane an elongation of the drop shape is occurring in the vertical direction to yield approximately prolate ellipsoid. Furthermore, a settling solid sphere changes greatly the shape of a rising bubble.

As expected, when two fluid particles are moving in opposite directions and the separation distance between them is small enough the so-called "dimple" is coming into view.

The present analysis shows that when the interfacial tension decreases the degree of the drop deformations increases and for large interfacial tension the drop shape is nearly spherical.

Further tasks are following:

i) Application of the method to the problem of a slow motion of a drop (bubble) towards a deformable fluid-liquid interface.

ii) Extension of the method to the three-dimensional problem of a quasisteady movement of a fluid particle parallel to a deformable fluid-liquid interface.

Work on these tasks is currently in progress.

REFERENCES

BART, E. 1968 The slow unsteady settling of a fluid sphere towards a flat fluid interface. *Chem. Engng. Sci.* **23**, 193–210.

BERDAN, C. & LEAL, L. G. 1982 Motion of a sphere in the presence of a deformable interface. 1. Perturbation of the interface from flat: the effects on drag and torque. *J. Colloid Interface Sci.* **87**, 62–80.

BRENNER, H. 1971 Dynamics of neutrally buoyant particles in low Reynolds number flows. *Prog. Heat Mass Transfer* **6**, 509–574.

BRENNER, H. 1961 The slow motion of a sphere through a viscous fluid towards a plane surface. *Chem. Engng. Sci.* **16**, 242–251.

CHAFFEY, C. E., BRENNER, H., & MASON, S. G. 1965 Particle motions in sheared suspensions. XVII Deformation and migration of liquid drops. *Rheologica-Acta* **4**, 56–63.

COX, R. G. 1969 The deformation of a drop in a general time dependent fluid flow. *J. Fluid Mech.* **37**, 601–623.

HABER, S., HETSRONI, G. & SOLAN, A. 1973 On the low Reynolds number motion of two droplets. *Int. J. Multiphase Flow* **1**, 57–71.

HADAMARD, J. S., 1911 *Compt. Rend.* **152**, 1735.

LEE, S. H., CHADWICK, R. S. & LEAL, L. G. 1979 Motion of a sphere in the presence of a plane interface. Part. 1. An approximate solution by generalization of the method of Lorentz. *J. Fluid Mech.* **93**, 705–726.

LEE, S. H. & LEAL, L. G. 1980 Motion of a sphere in the presence of a plane interface. Part. 2. An exact solution in bipolar coordinates. *J. Fluid Mech.* **98**, 192–224.

LEE, S. H. & LEAL, L. G. 1982 The motion of a sphere in the presence of a deformable interface. 2. A numerical study of the translation of a sphere normal to an interface. *J. Colloid Interface Sci.* **87**, 81–106.

LIN, C. Y. & SLATTERY, J. C. 1982 AIChE **28**, 147–156.

LORENTZ, H. A. 1907 *Abh. theoret. Phys.* **1**, 23.

MAUDE, A. D. 1961 End effects in a falling-sphere viscometer. *Br. J. Appl. Phys.* **12**, 293–295.

O'NEILL, M. E. & RANGER, K. B., 1983 The approach of a sphere to an interface. *Phys. Fluids* **26**(8), 2035–2042.

PAN, F. Y. & ACRIVOS, A. 1968 Shape of a drop or bubble at low Reynolds number. *IEC Fundamentals* 7(2), 227–232.

RUSHTON, E. & DAVIES, G. A., 1973 The motion of liquid droplets in settling of two fluid spheres along their line of centres. *Appl. Sci. Res.* **28**, 37–61.

RUSHTON, E. & DAVIES, G. A. 1978 The slow motion of two spherical particles along their line of centres. *Int. J. Multiphase Flow* **4**, 357–381.

RYBCZYNSKI, W., 1911 *Bull. Acad. Sci. Cracovic A* **40.**

SCRIVEN, L. E. Dynamics of a fluid interface: Equation of motion for Newtonian surface fluids. *Chem. Engng. Sci.* **12**, 98–108.

STIMSON, M. & JEFFERY, G. B. 1926 The motion of two spheres in viscous fluid. *Proc. Roy. Soc. A* **111**, 110.

TAYLOR, C. I. 1932 The viscosity of a fluid containing small drops of another fluid. *Proc. Roy. Soc. A* **138**, 41–48.

TAYLOR, G. I. 1934 The formation of emulsions in definable fields of flow. *Proc. Roy. Soc. A* **146**, 501–523.

TAYLOR, T. D. & ACRIVOS, A. 1964 On the deformation and drag of *J. Fluid Mech.* **18**, 466–476.